

# RANDOM REGULARIZATION OF BROWN SPECTRAL MEASURE

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**ABSTRACT.** We generalize a recent result of Haagerup; namely we show that a convolution with a standard Gaussian random matrix regularizes the behavior of Fuglede–Kadison determinant and Brown spectral distribution measure. In this way it is possible to establish a connection between the limit eigenvalues distributions of a wide class of random matrices and the Brown measure of the corresponding limits.

## 1. INTRODUCTION

The problem of determining the joint distribution of the eigenvalues of a random matrix  $A^{(N)} \in \mathcal{M}_N(\mathcal{L}^{\infty-}(\Omega))$  with a given distribution of entries is usually very hard and has explicit solutions only for very limited cases. A partial solution of this problem is to consider the limit  $N \rightarrow \infty$  and to hope that it is an easier problem than the original one.

Questions of this type can often be solved by Voiculescu’s theory of free probability [VDN]: if a sequence  $(A^{(N)})$  of random matrices converges in  $\star$ -moments to some  $x$ , where  $x$  is an element of a non-commutative probability space  $(\mathcal{A}, \phi)$ , then some properties of matrices  $A^{(N)}$  (e.g. independence of some entries) imply useful properties of the limit object  $x$  (e.g. some kind of freeness). If random matrices  $A^{(N)}$  are normal then the empirical distribution of their eigenvalues converges to the spectral measure of  $x$ . This approach turned out to be successful in determining the limit eigenvalues distribution in many cases (see e.g. [Shl]).

The situation is much more complicated if random matrices  $A^{(N)}$  and the limit object  $x$  are not normal. In this case the spectral measure of  $x$  has to be replaced by a more complicated object, namely by the Brown measure of  $x$ . The problem of determining the Brown measure of a given operator is still difficult, but it is much easier than the original question about the eigenvalues of a random matrix [Lar, HL, BL]. However, since the Brown measure does not behave in a continuous way with respect to the topology given by  $\star$ -moments, the distribution

of eigenvalues of  $A^{(N)}$  does not always converge to the Brown measure of  $x$ .

Surprisingly, in many known cases when we consider a “reasonable” or “generic” sequence of random matrices the sequence of their Brown measures converges to the Brown measure of the limit (cf [BL]). In these cases, however, the convergence was proved in this way, that the distribution of eigenvalues of  $A^{(N)}$  was calculated by ad hoc methods and nearly by an accident it turned out to converge to the Brown measure of  $x$ . Therefore one of the most interesting problems in the theory of random matrices is to relate the asymptotic distribution of eigenvalues of a sequence of random matrices with the Brown measure of the limit.

In this article we show that it is possible to add a small random correction to a sequence of random matrices  $(A^{(N)})$  which converges in  $\star$ -moments almost surely to some element  $x \in (\mathcal{A}, \phi)$  in such a way that the new corrected sequence still converges to  $x$  and that furthermore the empirical eigenvalues distributions of the new sequence converge to the Brown measure of  $x$  almost surely. A slightly different version of this result was proved recently by Haagerup [Haa] and plays a key role in his proof of existence of invariant subspaces for a large class of operators. The random correction used by him is a matrix Cauchy distribution, the first moment of which is unbounded, what makes it unpleasant for applications. For this reason Haagerup’s estimates of the correction were in the  $\mathcal{L}^p$  norm with  $0 < p < 1$ . The random correction considered in this article has a nicer form of a Gaussian random matrix and for this reason we are able to find better estimates for the correction, namely in the operator norm.

The Gaussianity of the correction allows us also to find the limit empirical eigenvalues distribution of a wide class of random matrices, which include both the well-known examples of the matrix  $G^{(N)}$  with suitably normalized independent Gaussian entries (the limit eigenvalues distribution was computed by Ginibre [Gin] in the sense of density of eigenvalues, the almost sure convergence of empirical distributions was proved in unpublished notes of Silverstein and later in more generality by Bai [Bai]; the Brown measure of the limit was computed by Haagerup and Larsen [Lar, HL]), the so-called elliptic ensemble (the limit eigenvalues distribution was computed by Petz and Hiai [PH] and the Brown measure of the limit was computed by Haagerup and Larsen [Lar, HL]) and new examples for which the eigenvalues distribution was not known before and which are of the form  $G^{(N)} + A^{(N)}$ , where entries of  $G^{(N)}$  and  $A^{(N)}$  are independent (the Brown measure of the limit of such matrices was computed by Biane and Lechner [BL]).

Results of this article can be also applied [Sni] to show that  $DT$  operators (which were introduced recently by Dykema and Haagerup [DH]) maximize microstate free entropy [Vo2] among all operators having fixed Brown measure and the second moment.

Our method bases on the observation that if a sequence of random matrices  $A^{(N)}$  converges in  $\star$ -moments to  $x$  then the Fuglede–Kadison determinants  $\Delta(A^{(N)})$  converge to  $\Delta(x)$  as well if we are able to find some bottom bounds for the smallest singular values of  $A^{(N)}$ . Since the random correction considered in this article is given by a certain matrix-valued Brownian motion, hence we are able to write a system of stochastic differential equations fulfilled by the singular values. Unfortunately, finding an exact analytic solution to a non-linear stochastic differential equation is very difficult. We deal with this problem by proving a certain monotonicity property of our equations and hence we are able to find appropriate bottom estimates for the singular values.

## 2. PRELIMINARIES

**2.1. Non-commutative probability spaces.** A non-commutative probability space is a pair  $(\mathcal{A}, \phi)$ , where  $\mathcal{A}$  is a  $C^*$ -algebra and  $\phi$  is a normal, faithful, tracial state on  $\mathcal{A}$ . Elements of  $\mathcal{A}$  will be referred to as non-commutative random variables and state  $\phi$  as expectation value. The distribution of  $x \in \mathcal{A}$  is the collection of all its  $\star$ -moments  $(\phi(x^{s_1} \cdots x^{s_n}))$ , where  $s_1, \dots, s_n \in \{1, \star\}$ .

**2.2. Fuglede–Kadison determinant.** Let a non-commutative probability space  $(\mathcal{A}, \phi)$  be given. For  $x \in \mathcal{A}$  we define its Fuglede–Kadison determinant  $\Delta(x)$  by (cf [FK])

$$\Delta(x) = \exp [\phi(\ln |x|)].$$

**2.3. Brown measure.** Let a non-commutative probability space  $(\mathcal{A}, \phi)$  be given. For  $x \in \mathcal{A}$  we define its Brown measure [Bro] to be the Schwartz distribution on  $\mathbb{C}$  given by

$$\mu_x = \frac{1}{2\pi} \left( \frac{\partial^2}{\partial a^2} + \frac{\partial^2}{\partial b^2} \right) \ln \Delta[x - (a + bi)].$$

One can show that in fact  $\mu_x$  is a positive probability measure on  $\mathbb{C}$ .

*Example.* The Brown measure of a normal operator has a particularly easy form; let  $x \in \mathcal{A}$  be a normal operator and let  $E$  denote its spectral measure:

$$x = \int_{\mathbb{C}} z \, dE(z).$$

Then the Brown measure of  $x$  is given by

$$\mu_x(X) = \phi[E(X)]$$

for every Borel set  $X \subseteq \mathbb{C}$  and the following holds:

$$\phi[x^k(x^*)^l] = \int_{\mathbb{C}} z^k \bar{z}^l d\mu_x(z).$$

**2.4. Random matrices.** We have that  $(\mathcal{M}_N, \text{tr}_N)$  is a non-commutative probability space, where  $\mathcal{M}_N$  denotes the set of all complex-valued  $N \times N$  matrices and  $\text{tr}_N$  (which for simplicity will be also denoted by  $\text{tr}$ ) is the normalized trace on  $\mathcal{M}_N$  given by

$$\text{tr}_N A = \frac{1}{N} \text{Tr } A \quad \text{for } A \in \mathcal{M}_N,$$

and  $\text{Tr}$  denotes the standard trace.

The below simple example shows that for finite matrices the Fuglede–Kadison determinant  $\Delta$  and the usual determinant  $\det$  are closely related and gives heuristical arguments that for every Borel set  $X \subset \mathbb{C}$  the Brown measure  $\mu_x(X)$  provides information on the joint “dimension” of “eigenspaces” corresponding to  $\lambda \in X$ .

**Proposition 1.** *The Fuglede–Kadison determinant of a matrix  $A \in \mathcal{M}_N$  with respect to a normalized trace  $\text{tr}$  is given by*

$$\Delta(A) = \sqrt[N]{|\det A|}.$$

*The Brown measure of a matrix  $A \in \mathcal{M}_N$  with respect to the state  $\text{tr}$  is a probability counting measure*

$$\mu_A = \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i},$$

*where  $\lambda_1, \dots, \lambda_N$  are the eigenvalues of  $A$  counted with multiples.*

In the following we will be interested in studying the random measure  $\omega \mapsto \mu_{A(\omega)}$  for a random matrix  $A \in \mathcal{M}_N(\mathcal{L}^{\infty-}(\Omega))$ . This random measure is called the empirical distribution of eigenvalues.

We will use the following convention: we say that random matrices  $A, B \in \mathcal{M}_N(\mathcal{L}^{\infty-}(\Omega))$  are independent if the family of entries of  $A$  and the family of entries of  $B$  are independent.

**2.5. Convergence of  $\star$ -moments.** Let a sequence  $A^{(N)} \in \mathcal{M}_N(\mathcal{L}^{\infty-}(\Omega))$  of random matrices, a non-commutative probability space  $(\mathcal{A}, \phi)$  and  $x \in \mathcal{A}$  be given. We say that the sequence  $A^{(N)}$

converges to  $x$  in  $\star$ -moments almost surely if for every  $n \in \mathbb{N}$  and  $s_1, \dots, s_n \in \{1, \star\}$  we have that

$$\lim_{N \rightarrow \infty} \operatorname{tr}_N [(A^{(N)})^{s_1} \dots (A^{(N)})^{s_n}] = \phi(x^{s_1} \dots x^{s_n})$$

holds almost surely.

Let a sequence  $A^{(N)} \in \mathcal{M}_N(\mathcal{L}^{\infty-}(\Omega))$  of random matrices, a non-commutative probability space  $(\mathcal{A}, \phi)$  and  $x \in \mathcal{A}$  be given. We say that the sequence  $A^{(N)}$  converges to  $x$  in expected  $\star$ -moments if for every  $n \in \mathbb{N}$  and  $s_1, \dots, s_n \in \{1, \star\}$  we have that

$$\lim_{N \rightarrow \infty} \mathbb{E} \operatorname{tr}_N [(A^{(N)})^{s_1} \dots (A^{(N)})^{s_n}] = \phi(x^{s_1} \dots x^{s_n}).$$

**2.6. Discontinuity of Fuglede–Kadison determinant and Brown measure.** One of the greatest difficulties connected with the Fuglede–Kadison determinant and Brown spectral distribution measure is that—as we shall see in the following example—these two objects do not behave in a continuous way with respect to the topology given by convergence of  $\star$ -moments.

We say that  $u \in \mathcal{A}$  is a Haar unitary if  $u$  is unitary and  $\phi(u^k) = \phi((u^*)^k) = 0$  for every  $k = 1, 2, \dots$ . It is not difficult to see that the sequence  $(\Xi^{(N)})$  converges in  $\star$ -moments to the Haar unitary, where  $\Xi^{(N)}$  is an  $N \times N$  nilpotent matrix

$$(1) \quad \Xi^{(N)} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}.$$

Every matrix  $\Xi^{(N)}$  has the determinant equal to 0, while the Haar unitary has the Fuglede–Kadison determinant equal to 1; every matrix  $\Xi^{(N)}$  has the Brown measure equal to  $\delta_0$ , while the Brown measure of the Haar unitary is the uniform measure on the unit circle  $\{z \in \mathbb{C} : |z| = 1\}$ .

The reason for the discontinuity of Fuglede–Kadison determinant is that the logarithm is not bounded from below on any interval  $[0, t]$ . However, since it is bounded from above, Fuglede–Kadison determinant is upper-semicontinuous.

**Lemma 2.** *Let  $A^{(N)}$  be a sequence of random matrices which converges in  $\star$ -moments to a non-commutative random variable  $x$  almost surely. Then for every  $\lambda \in \mathbb{C}$*

$$\limsup_{N \rightarrow \infty} \operatorname{tr} \ln |A^{(N)} - \lambda| \leq \ln \Delta(x - \lambda)$$

holds almost surely.

Let  $A^{(N)}$  be a sequence of random matrices which converges in expected  $\star$ -moments to a non-commutative random variable  $x$ . Then for every  $\lambda \in \mathbb{C}$  we have

$$\limsup_{N \rightarrow \infty} \mathbb{E} \operatorname{tr} \ln |A^{(N)} - \lambda| \leq \ln \Delta(x - \lambda).$$

*Proof.* For each  $\epsilon > 0$  there exists an even polynomial  $Q$  such that

$$\ln r \leq Q(r) \quad \text{for every } r > 0$$

and

$$Q(r) \leq \frac{\ln(r^2 + \epsilon)}{2} \quad \text{for every } 0 \leq r \leq \|x\|.$$

Hence

$$\operatorname{tr} \ln |A^{(N)} - \lambda| \leq \operatorname{tr} Q(|A^{(N)} - \lambda|).$$

The right-hand side converges almost surely (resp. in the expectation value) to  $\phi[Q(|x - \lambda|)] \leq \phi\left(\frac{\ln(r^2 + \epsilon)}{2}\right)$ . By taking the limit  $\epsilon \rightarrow 0$  both parts of the lemma follow.  $\square$

**2.7. Gaussian random matrices.** We say that a random matrix

$$G^{(N)} = (G_{ij}^{(N)})_{1 \leq i, j \leq N} \in \mathcal{M}_N(\mathcal{L}^{\infty-}(\Omega))$$

is a standard Gaussian random matrix if

$$(\Re G_{ij}^{(N)})_{1 \leq i, j \leq N}, (\Im G_{ij}^{(N)})_{1 \leq i, j \leq N}$$

are independent Gaussian variables with mean zero and variance  $\frac{1}{2N}$ .

We say that

$$M^{(N)} : \mathbb{R}_+ \rightarrow \mathcal{M}_N(\mathcal{L}^{\infty-}(\Omega)), \quad M^{(N)}(t) = (M_{ij}^{(N)}(t))_{1 \leq i, j \leq N}$$

is a standard matrix Brownian motion if

$$(\Re M_{ij}^{(N)})_{1 \leq i, j \leq N}, (\Im M_{ij}^{(N)})_{1 \leq i, j \leq N}$$

are independent Brownian motions which are normalized in such a way that the variance is given by

$$\mathbb{E}(\Re M_{ij}^{(N)}(t))^2 = \mathbb{E}(\Im M_{ij}^{(N)}(t))^2 = \frac{t}{2N}.$$

**2.8. Circular element.** There are many concrete characterizations of the Voiculescu's circular element  $c$  [VDN] but we will use the following implicit definition. One can show that the sequence  $G^{(N)}$  converges both in expected  $\star$ -moments and in  $\star$ -moments almost surely to a certain non-commutative random variable  $c$  [Vo1, Tho].

**2.9. Freeness.** Let  $(\mathcal{A}, \phi)$  be a non-commutative probability space and let  $(\mathcal{A}_i)_{i \in I}$  be a family of unital  $\star$ -subalgebras of  $\mathcal{A}$ . We say that the algebras  $(\mathcal{A}_i)_{i \in I}$  are free if

$$\phi(x_1 x_2 \cdots x_n) = 0$$

holds for every  $n \geq 1$ , every  $i_1, i_2, \dots, i_n \in I$  such that  $i_1 \neq i_2, i_2 \neq i_3, \dots, i_{n-1} \neq i_n$ , and every  $x_1 \in \mathcal{A}_{i_1}, \dots, x_n \in \mathcal{A}_{i_n}$  such that  $\phi(x_1) = \dots = \phi(x_n) = 0$  (cf [VDN]).

Let  $(X_i)_{i \in I}$  be a family of subsets of  $\mathcal{A}$ . We say that sets  $X_i$  are free if unital  $\star$ -algebras  $(\text{Alg}\{X_i, X_i^*\})_{i \in I}$  are free.

### 3. SINGULAR VALUES OF MATRIX BROWNIAN MOTIONS

Let  $A \in \mathcal{M}_N(\mathcal{L}^{\infty-}(\Omega))$  be a given random matrix and  $M^{(N)}$  be a standard matrix Brownian motion such that  $A$  and  $M^{(N)}$  are independent. For  $t \geq 0$  we define a random matrix  $A_t$  by

$$A_t(\omega) = A(\omega) + M^{(N)}(t, \omega).$$

It should be understood that matrix Brownian motions  $(M^{(N)})_{N=1,2,\dots}$  are independent.

If we are interested in  $A_t$  for only one value of  $t \geq 0$  we can express  $A_t$  as follows:

$$A_t(\omega) = A(\omega) + \sqrt{t} G^{(N)}(\omega),$$

where  $G^{(N)}$  is a standard Gaussian random matrix such that  $A$  and  $G^{(N)}$  are independent.

If  $x \in \mathcal{A}$  is a non-commutative random variable, we can always extend the algebra  $\mathcal{A}$  and find  $c \in \mathcal{A}$  such that  $\{x, x^*\}$  and  $\{c, c^*\}$  are free and  $c$  is a circular element [VDN]. We will denote

$$x_t = x + \sqrt{t} c.$$

**Proposition 3.** *If sequence of random matrices  $|A^{(N)}|^2$  converges in  $\star$ -moments to  $|x|^2$  almost surely then for every  $t \geq 0$  the sequence  $|A_t^{(N)}|^2$  converges in  $\star$ -moments to  $|x_t|^2$  almost surely.*

*If sequence of non-random matrices  $|A^{(N)}|^2$  converges in  $\star$ -moments to  $|x|^2$  then for every  $t \geq 0$  the sequence  $|A_t^{(N)}|^2$  converges in expected  $\star$ -moments to  $|x_t|^2$ .*

*Proof.* The first part of the propositions follows under additional assumption that  $\sup_N \|A^{(N)}\| < \infty$  almost surely from recent results of Hiai and Petz [HP]. For the general case observe that since for all unitary matrices  $U, V \in \mathcal{M}_N$  and  $n \in \mathbb{N}$  the distributions of random variables  $\text{tr}|A^{(N)} + \sqrt{t}G^{(N)}|^{2n}$  and  $\text{tr}|A^{(N)} + \sqrt{t}UG^{(N)}V|^{2n} = \text{tr}|V^*A^{(N)}U + \sqrt{t}G^{(N)}|^{2n}$  coincide, hence it is enough to prove the

first part under assumption that every matrix  $A^{(N)}$  is almost surely diagonal. The method of Thorbjørnsen can be generalized to this case [Tho].

The second part of the proposition was proved by Voiculescu [Vo1].  $\square$

For any  $t \geq 0$  and  $\omega \in \Omega$  let  $\lambda_1(t, \omega) \geq \dots \geq \lambda_N(t, \omega)$  denote singular values of the matrix  $A_t(\omega)$ .

In Section 5.1 we derive stochastic differential equations for  $\lambda_1, \dots, \lambda_N$  using similar methods to those of Chan [Cha] and obtain

$$(2) \quad d\lambda_i(t) = \Re(dB_{ii}) + \frac{dt}{2\lambda_i} \left( 1 - \frac{1}{2N} + \sum_{j \neq i} \frac{\lambda_i^2 + \lambda_j^2}{N(\lambda_i^2 - \lambda_j^2)} \right),$$

where  $B$  is a standard matrix Brownian motion.

**Theorem 4.** *Let  $A^{(1)}$  and  $A^{(2)}$  be non-random matrices of the same size,  $A^{(1)}, A^{(2)} \in \mathcal{M}_N$ . For  $n = 1, 2$  let  $s_1^{(n)} \geq \dots \geq s_N^{(n)}$  be the singular values of the matrix  $A^{(n)}$ . Suppose that for each  $1 \leq k \leq N$  we have  $s_k^{(1)} < s_k^{(2)}$ .*

*Then for every  $t \geq 0$  there exists a probability space  $(\Omega, \mathcal{B}, P)$  and random matrices  $G^{(1)}, G^{(2)} \in \mathcal{M}_N(\mathcal{L}^{\infty-}(\Omega))$  such that each matrix  $G^{(i)}$  is a standard Gaussian random matrix (but matrices  $G^{(1)}$  and  $G^{(2)}$  might be dependent) and such that*

$$\mathrm{tr} f(|A^{(1)} + \sqrt{t} G^{(1)}(\omega)|) \leq \mathrm{tr} f(|A^{(2)} + \sqrt{t} G^{(2)}(\omega)|)$$

*holds for every  $\omega \in \Omega$  and every nondecreasing function  $f : \mathbb{R} \rightarrow \mathbb{R}$ .*

*Proof.* Let us consider a probability space  $(\Omega, \mathcal{B}, P)$ , a standard matrix Brownian motion  $B : \mathbb{R}_+ \rightarrow \mathcal{M}_N(\mathcal{L}^{\infty-}(\Omega))$  and for each  $n \in \{1, 2\}$  we find the solution of the system of stochastic differential equations

$$(3) \quad d\lambda_i^{(n)} = \Re(dB_{ii}) + \frac{dt}{2\lambda_i^{(n)}} \left( 1 - \frac{1}{2N} + \sum_{j \neq i} \frac{(\lambda_i^{(n)})^2 + (\lambda_j^{(n)})^2}{N((\lambda_i^{(n)})^2 - (\lambda_j^{(n)})^2)} \right).$$

together with initial conditions

$$\lambda_i^{(n)}(0, \omega) = s_i^{(n)}.$$

We have that for each  $t > 0$  and  $n \in \{1, 2\}$  the joint distribution of random variables  $\lambda_i^{(n)}(t)$ ,  $i = 1, \dots, N$  coincides with the joint distribution of singular values of the random matrix  $A_t^{(n)}$ .



The theorem will follow from the following stronger statement: for almost every  $\omega$  and every  $t \geq 0$  we have

$$(4) \quad \lambda_i^{(1)}(t, \omega) < \lambda_i^{(2)}(t, \omega) \quad \text{for every } 1 \leq i \leq N.$$

From Eq. (3) it follows that for almost every  $\omega \in \Omega$  we have that  $\lambda^{(1)} - \lambda^{(2)}$  has a continuous derivative (see Section 5.1.2). For a fixed  $\omega \in \Omega$  let  $t_0$  be the smallest  $t \geq 0$  such that (4) does not hold. Trivially we have  $t_0 > 0$ . There exists an index  $j$  such that  $\lambda_j^{(1)}(t_0) = \lambda_j^{(2)}(t_0) =: \lambda_j$  and for every  $i$  we have  $\lambda_i^{(1)}(t_0) \leq \lambda_i^{(2)}(t_0)$ . Eq. (3) gives us

$$(5) \quad \left. \frac{d}{dt} \left( \lambda_j^{(1)}(t) - \lambda_j^{(2)}(t) \right) \right|_{t=t_0} = \sum_{k \neq j} \frac{\lambda_j \left( (\lambda_k^{(1)})^2 - (\lambda_k^{(2)})^2 \right)}{N \left( \lambda_j^2 - (\lambda_k^{(1)})^2 \right) \left( \lambda_j^2 - (\lambda_k^{(2)})^2 \right)}.$$

It is easy to see that if there exists at least one index  $1 \leq k \leq N$  such that  $\lambda_k^{(1)}(t_0) \neq \lambda_k^{(2)}(t_0)$  then

$$\left. \frac{d}{dt} \left( \lambda_j^{(1)} - \lambda_j^{(2)} \right) \right|_{t=t_0} < 0,$$

so it follows that for small  $d > 0$  we have  $\lambda_j^{(1)}(t) - \lambda_j^{(2)}(t) > 0$  for  $t_0 - d < t < t_0$ . This contradicts the minimality of  $t_0$ .

We define  $\delta(t) = \lambda_i^{(1)}(t) - \lambda_i^{(2)}(t)$ . If we replace in (5)  $\lambda_i^{(2)}$  by  $\lambda_i^{(1)} - \delta_i$  then it becomes a system of non-stochastic ordinary differential equations for  $\delta_i$ . If for all indexes  $1 \leq i \leq N$  we have  $\lambda_i^{(1)}(t_0) = \lambda_i^{(2)}(t_0)$  then  $\delta_i(t_0) = 0$  and the solution exists and is unique in some (backward) interval. This contradicts the minimality of  $t_0$ .  $\square$

**Proposition 5.** *If  $A^{(N)} \in \mathcal{M}_N(\mathcal{L}^{\infty-}(\Omega))$  is a random matrix and  $\lambda \in \mathbb{C}$  then the function  $\mathbb{R}_+ \ni t \mapsto \mathbb{E} \operatorname{tr} \ln |A_t^{(N)} - \lambda|$  is nondecreasing.*

*For  $x \in \mathcal{A}$  and  $\lambda \in \mathbb{C}$  we have that the function  $\mathbb{R}_+ \ni t \mapsto \ln \Delta(x_t)$  is nondecreasing and*

$$(6) \quad \lim_{t \rightarrow 0^+} \ln \Delta(x_t) = \ln \Delta(x).$$

*Proof.* We can regard  $\mathcal{M}_N$  as a  $2N^2$ -dimensional real Euclidean space equipped with a scalar product  $\langle m, n \rangle = \Re \operatorname{Tr} mn^*$ . As usually we define the Laplacian to be  $\nabla^2 = \sum_{1 \leq k \leq 2N^2} D_{v_k}^2$ , where  $v_1, \dots, v_{2N^2}$  is the orthonormal basis of this space and  $D_v$  is a derivative operator in direction  $v$ .

Notice that  $\ln |\det A| = \Re \ln \det A$ . We can regard  $\det A$  as a holomorphic function of  $N^2$  complex variables (=entries of the matrix). On the other hand it is a known-fact that if  $f(z_1, \dots, z_k)$  is a holomorphic

function then the Laplacian of its logarithm is a positive measure. This and Itô formula imply the first part of the proposition.

For the second part we construct a sequence  $(A^{(N)})$ , where  $A^{(N)} \in \mathcal{M}_N$ , such that  $A^{(N)}$  converges in  $\star$ -moments to  $|x|$  and such that  $\lim_{N \rightarrow \infty} \operatorname{tr} \ln(A^{(N)}) = \ln \Delta(x)$  and apply Lemma 2 for the sequence  $A_t^{(N)}$ . This shows that  $\mathbb{R}_+ \ni t \mapsto \ln \Delta(x_t)$  is nondecreasing.

On the other hand the inequality

$$\limsup_{t \rightarrow 0^+} \ln \Delta(x_t) \leq \ln \Delta(x)$$

can be proved similarly as in Lemma 2.  $\square$

#### 4. THE MAIN RESULT

**Theorem 6.** *Let  $A^{(N)} \in \mathcal{M}_N(\mathcal{L}^{\infty-}(\Omega))$  be a sequence of random matrices such that  $A^{(N)}$  converges in  $\star$ -moments to a non-commutative random variable  $x$  almost surely.*

*For every  $t > 0$  we have that the sequence of empirical distributions  $\mu_{A_t^{(N)}(\omega)}$  converges in the weak topology to  $\mu_{x_t}$  almost surely.*

*There exists a sequence  $(t_N)$  of positive numbers such that  $\lim_{N \rightarrow \infty} t_N = 0$  and the sequence of empirical distributions  $\mu_{A_{t_N}^{(N)}(\omega)}$  converges in the weak topology to  $\mu_x$  almost surely.*

**Theorem 7.** *Let  $(A^{(N)})$  be a sequence of non-random matrices ( $A^{(N)} \in \mathcal{M}_N$ ) which converges in  $\star$ -moments to a non-commutative random variable  $x \in \mathcal{A}$ , where  $(\mathcal{A}, \phi)$  is a non-commutative probability space.*

*There exists a sequence  $(\tilde{A}^{(N)})$  of non-random matrices such that the distributions of eigenvalues  $\mu_{\tilde{A}^{(N)}}$  converge weakly to  $\mu_x$  and*

$$\lim_{N \rightarrow \infty} \|A^{(N)} - \tilde{A}^{(N)}\| = 0,$$

*where  $\|\cdot\|$  denotes the operator norm of a matrix.*

As an illustration to the above theorems we present on Fig. 1—4 results of a computer experiment; we plotted eigenvalues of the nilpotent matrix  $\Xi_N$  from Eq. (1) with a random Gaussian correction. The size of the matrices was  $N = 100$ ; with dashed line we marked the spectrum of the Haar unitary, which is the circle of radius 1 centered in 0. We recall that the sequence  $(\Xi^{(N)})$  converges in  $\star$ -moments to the Haar unitary. As one can see if the random correction is too small then the eigenvalues of the corrected matrix behave like the eigenvalues of  $\Xi^{(N)}$  and if the random correction is too big then the eigenvalues of the corrected matrix are dispersing on the plane.

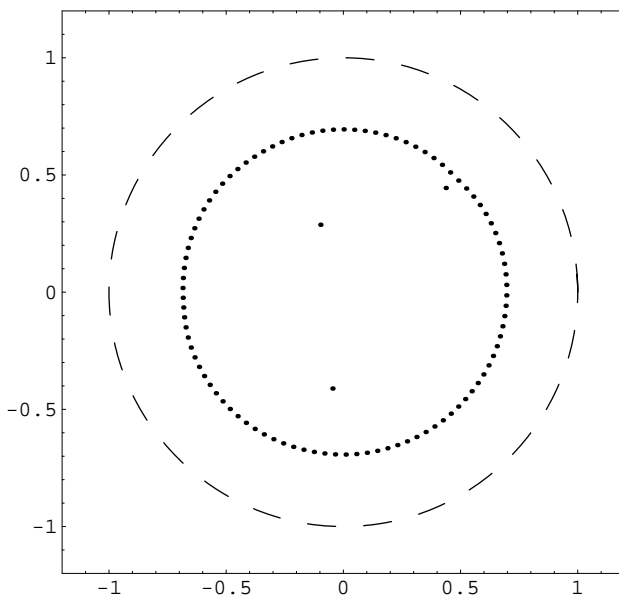


FIGURE 1. Sample eigenvalues of a random matrix  $\Xi^{(N)} + \sqrt{t}G^{(N)}$  for  $N = 100$  and  $t = 10^{-100}$ .

An interesting problem for future research is for a given sequence  $A^{(N)}$  of random matrices which converges in  $\star$ -moments to  $x$  and fixed  $N$  to determine the optimal value of  $t_N$  for which the measure  $\mu_{A_{t_N}^{(N)}}$  is the best approximation of  $\mu_x$ .

Before we present the proofs of these theorems we shall prove the following lemma.

**Lemma 8.** *Let  $A^{(N)}$  be as in Theorem 6. For every  $t > 0$  and every  $\lambda \in \mathbb{C}$  we have that*

$$\lim_{N \rightarrow \infty} \text{tr} \ln |A_t^{(N)} - \lambda| = \ln \Delta(x_t - \lambda)$$

*holds almost surely.*

*Proof.* Let us fix  $\lambda \in \mathbb{C}$ . For any  $\epsilon > 0$  we define functions on  $\mathbb{R}_+$

$$f_\epsilon(r) = \frac{\ln(r^2 + \epsilon)}{2},$$

$$g_\epsilon(r) = \ln r - \frac{\ln(r^2 + \epsilon)}{2}.$$

Each function  $f_\epsilon$  is well defined on  $[0, \infty)$  and  $f_\epsilon$  converges to the function  $t \mapsto \ln t$  pointwise as  $\epsilon$  tends to 0. Each function  $g_\epsilon$  is increasing and  $g_\epsilon$  converges pointwise to 0 as  $\epsilon$  tends to 0.

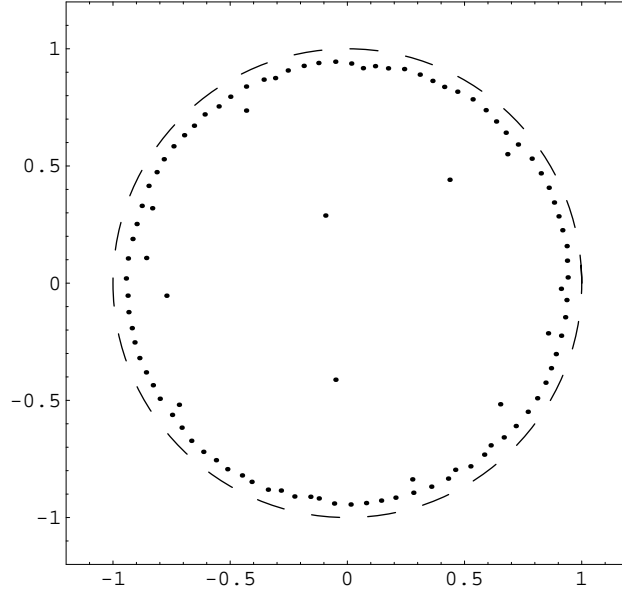


FIGURE 2. Sample eigenvalues of a random matrix  $\Xi^{(N)} + \sqrt{t}G^{(N)}$  for  $N = 100$  and  $t = 10^{-5}$ .

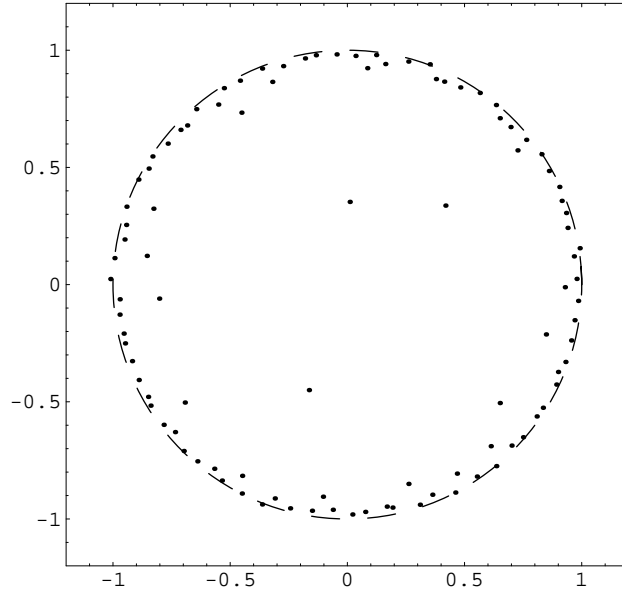


FIGURE 3. Sample eigenvalues of a random matrix  $\Xi^{(N)} + \sqrt{t}G^{(N)}$  for  $N = 100$  and  $t = 10^{-2}$ .

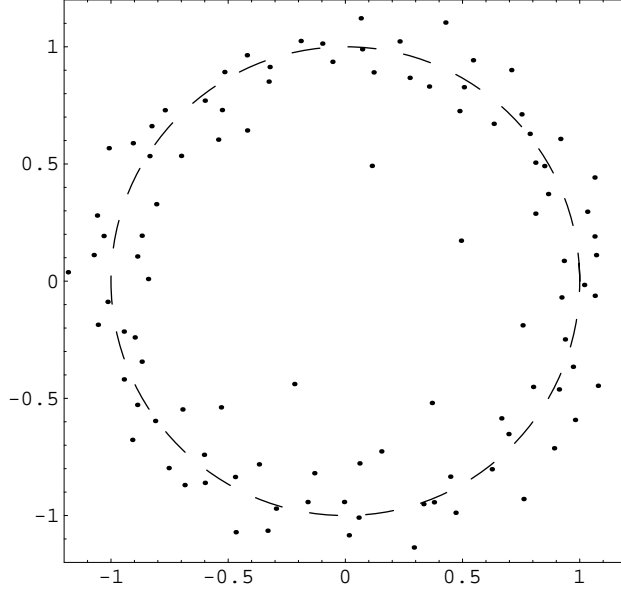


FIGURE 4. Sample eigenvalues of a random matrix  $\Xi^{(N)} + \sqrt{t}G^{(N)}$  for  $N = 100$  and  $t = 3 \cdot 10^{-1}$ .

Since the function  $f_\epsilon$  has a polynomial growth at infinity, therefore there exist even polynomials  $S(r)$  and  $Q(r)$  such that  $S(r) \leq f_\epsilon(r) \leq Q(r)$  holds for every  $r \geq 0$  and furthermore  $|S(r) - f_\epsilon(r)| < \epsilon$ ,  $|Q(r) - f_\epsilon(r)| < \epsilon$  hold for every  $0 \leq r \leq \|x_t\|$ .

We apply Theorem 4 for a pair of matrices 0 and  $A^{(N)} - \lambda$  and obtain a probability space  $(\Omega, \mathcal{B}, P)$  and Gaussian random matrices  $\tilde{G}^{(N)}, G^{(N)} \in \mathcal{M}_N(\mathcal{L}^{\infty-}(\Omega))$  such that

$$\mathrm{tr} g_\epsilon(|\sqrt{t} \tilde{G}^{(N)}|) \leq \mathrm{tr} g_\epsilon(|A_t^{(N)} - \lambda|)$$

holds for every  $\omega \in \Omega$ , where as usually  $A_t^{(N)} = A^{(N)} + \sqrt{t} G^{(N)}$ . For simplicity here and in the following we skip the obvious dependence of random variables on  $\omega$ .

We have that

$$\begin{aligned} \mathrm{tr} \ln |A_t^{(N)} - \lambda| &= \mathrm{tr} f_\epsilon(|A_t^{(N)} - \lambda|) + \mathrm{tr} g_\epsilon(|A_t^{(N)} - \lambda|) \geq \\ &\quad \mathrm{tr} S(|A_t^{(N)} - \lambda|) + \mathrm{tr} g_\epsilon(|\sqrt{t} \tilde{G}^{(N)}|) = \\ &\quad \mathrm{tr} S(|A_t^{(N)} - \lambda|) + \mathrm{tr} \ln(|\sqrt{t} \tilde{G}^{(N)}|) - \mathrm{tr} f_\epsilon(|\sqrt{t} \tilde{G}^{(N)}|) \geq \\ &\quad \mathrm{tr} S(|A_t^{(N)} - \lambda|) + \mathrm{tr} \ln(|\sqrt{t} \tilde{G}^{(N)}|) - \mathrm{tr} Q(|\sqrt{t} \tilde{G}^{(N)}|) =: X^{(N)}. \end{aligned}$$

Propositions 3 and 9 show that

$$\begin{aligned} \lim_{N \rightarrow \infty} X^{(N)} &= \phi(S(|x_t - \lambda|)) + \phi(\ln |\sqrt{t}c|) - \phi(Q(|\sqrt{t}c|)) \geq \\ &\quad \phi(\ln |x_t - \lambda|) + \phi(g_\epsilon(|\sqrt{t}c|)) - 2\epsilon \end{aligned}$$

holds almost surely. Hence by taking the limit  $\epsilon \rightarrow \infty$  we obtain that the inequality

$$\liminf_{N \rightarrow \infty} \operatorname{tr} \ln |A_t^{(N)} - \lambda| \geq \ln \Delta(x_t - \lambda)$$

holds almost surely.

The upper estimate

$$\limsup_{N \rightarrow \infty} \operatorname{tr} \ln |A_t^{(N)} - \lambda| \leq \ln \Delta(x_t - \lambda)$$

follows from Lemma 2 and Proposition 3, what finishes the proof.  $\square$

*Proof of Theorem 6.* For the proof of the first part of the theorem let  $K \subset \mathbb{C}$  be a compact set. In the following  $y$  will denote either  $x \in \mathcal{A}$  or the matrix  $A_t^{(N)}(\omega)$ . Let  $f \in C^2(K)$  be a smooth enough function with a compact support  $K \subset \mathbb{C}$ . From the definition of the Brown measure we have

$$\begin{aligned} \int_{\mathbb{C}} f(\lambda) d\mu_y(\lambda) &= \frac{1}{2\pi} \langle f(\lambda), \nabla^2 \ln \Delta(y - \lambda) \rangle = \\ &\quad \frac{1}{2\pi} \int_{\mathbb{C}} \ln \Delta(y - \lambda) \nabla^2 f(\lambda) d\lambda \end{aligned}$$

Since twice differentiable functions  $C^2(K)$  are dense in the set of all continuous functions  $C(K)$  therefore the almost certain convergence of measures  $\mu_{A_t^{(N)}(\omega)}$  in the weak topology to the measure  $\mu_x$  would follow if the sequence of functions  $\operatorname{tr} \ln |A_t^{(N)}(\omega) - \lambda|$  converges to the function  $\ln \Delta(x - \lambda)$  in the local  $\mathcal{L}^1$  norm almost surely. Therefore it would be sufficient to show that for almost every  $\omega \in \Omega$  we have (for simplicity here and in the following we skip the obvious dependence of random variables on  $\omega$ )

$$\lim_{N \rightarrow \infty} \int_K \left| \operatorname{tr} \ln |A_t^{(N)} - \lambda| - \ln \Delta(x_t - \lambda) \right| d\lambda = 0.$$

From Lemma 8 and the Fubini theorem follows that for almost every  $\omega \in \Omega$  we have

$$\lim_{N \rightarrow \infty} \operatorname{tr} \ln |A_t^{(N)} - \lambda| = \ln \Delta(x_t - \lambda)$$

for almost all  $\lambda \in K$ . Now it is sufficient to show that

$$(7) \quad \int_K \sup_N \left| \operatorname{tr} \ln |A_t^{(N)} - \lambda| \right| d\lambda + \int_K \left| \ln \Delta(x_t - \lambda) \right| d\lambda < \infty$$

holds almost surely in order to apply the majorized convergence theorem.

Note that  $\lambda \mapsto \log \Delta(x_t - \lambda)$  is subharmonic [Bro] and hence it is a local  $\mathcal{L}^1$  function; therefore we only need to find estimates for the first summand in (7).

Theorem 4 gives us that for almost every  $\omega \in \Omega$

$$\operatorname{tr} \ln |A_t^{(N)} - \lambda| \geq \operatorname{tr} \ln |\sqrt{t} \tilde{G}^{(N)}|$$

hence Proposition 9 implies that

$$\mathbb{E} \min \left( 0, \inf_N \operatorname{tr} \ln |A_t^{(N)} - \lambda| \right) \geq \mathbb{E} \min \left( 0, \inf_N \operatorname{tr} \ln |G^{(N)}| + \frac{\ln t}{2} \right)$$

is uniformly bounded from below over  $\lambda \in \mathbb{C}$ . From Fubini theorem follows that for almost every  $\omega \in \Omega$  we have

$$\int_K \min \left( 0, \inf_N \operatorname{tr} \ln |A_t^{(N)}(\omega) - \lambda| \right) d\lambda > -\infty.$$

From the simple inequality  $\log r < r^2$  which holds for every  $r > 0$  we have

$$\operatorname{tr} \ln |A_t^{(N)} - \lambda| < \operatorname{tr} |A_t^{(N)} - \lambda|^2 \leq \sqrt{\operatorname{tr} |A_t^{(N)}|^2 + \lambda^2}.$$

By Proposition 3 we have that  $\operatorname{tr} |A_t^{(N)}|^2$  converges almost surely, hence the family of functions  $K \ni \lambda \mapsto \operatorname{tr} \ln |A_t^{(N)} - \lambda|$  is almost surely uniformly bounded from above, what finishes the proof of the first part of the theorem.

From the first part of theorem follows that there exists a decreasing sequence  $(t_N)$  of positive numbers which converges to 0 and such that for any compact  $K \subset \mathbb{C}$

$$\lim_{N \rightarrow \infty} \int_K \left| \operatorname{tr} \ln |A_{t_N}^{(N)} - \lambda| - \ln \Delta(x_{t_N} - \lambda) \right| d\lambda = 0$$

holds almost surely.

Proposition 5 implies that the majorized convergence theorem can be applied (we recall that  $\lambda \mapsto \log \Delta(y - \lambda)$  is always a local  $\mathcal{L}^1$  function) hence

$$\lim_{N \rightarrow \infty} \int_K \left| \ln \Delta(x_{t_N} - \lambda) - \ln \Delta(x - \lambda) \right| d\lambda = 0.$$

The above two equations combine to give

$$\lim_{N \rightarrow \infty} \int_K \left| \operatorname{tr} \ln |A_{t_N}^{(N)} - \lambda| - \ln \Delta(x - \lambda) \right| d\lambda = 0$$

almost surely. The convergence of empirical distributions of eigenvalues follows now exactly as in the proof of the first part.  $\square$

*Proof of Theorem 7.* Let  $(t_N)$  be a sequence given by Theorem 6. Since  $\limsup_{N \rightarrow \infty} \|G^{(N)}\| < \infty$  almost surely [Gem], hence for almost every  $\omega \in \Omega$

$$\tilde{A}^{(N)} = A^{(N)} + \sqrt{t_N} G^{(N)}(\omega)$$

is the wanted sequence.  $\square$

## 5. TECHNICAL RESULTS

### 5.1. Derivation of the stochastic differential equation for $\lambda_i$ .

5.1.1. *Singular values as functions on  $\mathcal{M}_N$ .* In this subsection we are going to evaluate the first and the second derivative of the map

$$s : \mathcal{M}_N \ni m \mapsto (s_1(m), \dots, s_N(m)),$$

where  $s_1(m), \dots, s_N(m)$  denote the singular values of a matrix  $m$ .

The perturbation theory shows (cf chapter II.2 of [Kat]) that if  $D$  is a diagonal matrix with eigenvalues  $\nu_1, \dots, \nu_N$  such that  $\nu_i \neq \nu_j$  for all  $i \neq j$  and  $\Delta D$  is any matrix then the eigenvalues  $\nu'_1, \dots, \nu'_N$  of a matrix  $D + \Delta D$  are given by

$$\nu'_i = \nu_i + \Delta D_{ii} + \sum_{j \neq i} \frac{\Delta D_{ij} \Delta D_{ji}}{\nu_i - \nu_j} + O(\|\Delta D\|^3)$$

for small enough  $\|\Delta D\|$  and that the map  $\Delta D \mapsto (\nu'_1, \dots, \nu'_N)$  is  $C^2$  in some neighbourhood of 0.

It follows that if  $F$  is a diagonal matrix with positive eigenvalues  $s_1, \dots, s_N$ , and  $\Delta F \in \mathcal{M}_N$  is any matrix then the singular values  $s'_1, \dots, s'_N$  of  $F + \Delta F$  are given by

$$(8) \quad (s'_i)^2 = s_i^2 + 2s_i \Re \Delta F_{ii} + \sum_j |\Delta F_{ji}|^2 \\ + \sum_{j \neq i} \frac{s_i^2 |\Delta F_{ij}|^2 + 2s_i s_j \Re(\Delta F_{ij} \Delta F_{ji}) + s_j^2 |\Delta F_{ji}|^2}{s_i^2 - s_j^2} + O(\|\Delta F\|^3)$$

and that the map  $\Delta F \mapsto (s'_1, \dots, s'_N)$  is  $C^2$  on some neighbourhood of 0.

In the general case every matrix  $X$  can be written as  $X = U F V$ , where  $F$  is a positive diagonal matrix and  $U, V$  are unitaries. If the singular values of  $X$  are  $s_1, \dots, s_N$  then (8) gives us singular values  $s'_1, \dots, s'_N$  of the matrix  $X + \Delta X$ , where  $\Delta F$  is defined by  $\Delta F = V^* \Delta X U^*$ .



5.1.2. *Trajectories of the Brownian motion avoid singularities of  $s$ .* The set of singularities of the map  $s$ , namely

$$\{m \in \mathcal{M}_N : \det m = 0 \text{ or } s_i(m) = s_j(m) \text{ for some } i \neq j\},$$

is a manifold of codimension 2 and hence almost every trajectory of a matrix Brownian motion  $A_t$  will avoid this set. In this subsection we will present a rigorous proof of this statement.

For every  $\epsilon > 0$  we define a set

$$K_\epsilon = \left\{ m \in \mathcal{M}_N : \sum_i \ln s_i(m) \geq \epsilon, \sum_{i < j} \ln |s_i(m)^2 - s_j(m)^2| \geq \epsilon \right\}.$$

First of all, for any fixed  $\epsilon > 0$  we define a stopping time

$$T(\omega) = \min\{t \geq 0 : \ln |\det A_t(\omega)| \leq \epsilon\}$$

and a stopped Brownian motion

$$\tilde{A}_t(\omega) = A_{\min[t, T(\omega)]}(\omega).$$

In the proof of Proposition 5 we showed the function  $m \mapsto \ln |\det m|$  is subharmonic and hence  $t \mapsto \ln |\det \tilde{A}_t|$  is a submartingale. It follows that

$$(9) \quad \ln |\det A| \leq \mathbb{E} \ln |\det \tilde{A}_T| \leq \mathbb{E} \max(0, \ln |\det A_T|) + \ln \epsilon P(\omega \in \Omega : \ln |\det A_t(\omega)| \leq \epsilon \text{ for some } 0 \leq t \leq T).$$

Since the first summand on the right-hand side of the above inequality is clearly finite, it follows that

$$\lim_{\epsilon \rightarrow 0^+} P(\omega \in \Omega : |\det A_t(\omega)| \geq \epsilon \text{ for every } 0 \leq t \leq T) = 1.$$

Secondly, we consider a function on  $M_N(\mathbb{C})$  given by

$$(10) \quad m \mapsto \sum_{i < j} \ln \left| (s_i(m))^2 - (s_j(m))^2 \right|.$$

Formula (8) gives us first and second derivatives of the map  $m \mapsto (s_1(m), \dots, s_N(m))$  and allows us to find the Laplacian of the each summand in (10):

$$\frac{\nabla^2 \ln |s_i^2 - s_j^2|}{4} = \frac{s_i^2 + s_j^2}{(s_i^2 - s_j^2)^2} + \sum_{k \neq i, j} \frac{s_i^2 + s_k^2}{(s_i^2 - s_j^2)(s_i^2 - s_k^2)} - \frac{s_j^2 + s_k^2}{(s_i^2 - s_j^2)(s_j^2 - s_k^2)}.$$

It is not difficult to see that for every  $i, j, k$  all different we have

$$\left( \frac{s_i^2 + s_k^2}{(s_i^2 - s_j^2)(s_i^2 - s_k^2)} - \frac{s_j^2 + s_k^2}{(s_i^2 - s_j^2)(s_j^2 - s_k^2)} \right) + \left( \frac{s_j^2 + s_i^2}{(s_j^2 - s_k^2)(s_j^2 - s_i^2)} - \frac{s_k^2 + s_i^2}{(s_j^2 - s_k^2)(s_k^2 - s_i^2)} \right) + \left( \frac{s_k^2 + s_j^2}{(s_k^2 - s_i^2)(s_k^2 - s_j^2)} - \frac{s_i^2 + s_j^2}{(s_k^2 - s_i^2)(s_i^2 - s_j^2)} \right) = 0$$

and due to these cancellations

$$\nabla^2 \sum_{i < j} \ln |s_i^2 - s_j^2| = 4 \sum_{i < j} \frac{s_i^2 + s_j^2}{(s_i^2 - s_j^2)^2} > 0$$

holds. It follows that

$$t \mapsto \sum_{i < j} \ln \left| (s_i(\tilde{A}_t))^2 - (s_j(\tilde{A}_t))^2 \right|$$

is a submartingale and by similar arguments as in (9) we see that

$$\lim_{\epsilon \rightarrow 0^+} P(\omega \in \Omega : \sum_{i < j} \ln |s_i(m)^2 - s_j(m)^2| \geq \epsilon \text{ for every } 0 \leq t \leq T) = 1.$$

**5.1.3. Stochastic differential equation for  $\lambda_i$ .** We recall that every matrix  $m$  can be written as  $m = U(m)F(m)V(m)$ , where  $F(m)$  is a positive diagonal matrix and  $U(m), V(m)$  are unitaries. Let us define now a new matrix-valued stochastic process  $B$  given by a stochastic differential equation  $dB = (V(A_t))^*(dM)U(A_t)^*$ . It is easy to see that  $B$  is again a standard matrix Brownian motion.

For any fixed  $\epsilon > 0$  let us consider any  $C^2$  function  $\tilde{s} : \mathcal{M}_N \rightarrow \mathbb{R}^N$  such that  $\tilde{s}(m) = s(m)$  for every matrix  $m \in K_\epsilon$  and such that  $\|s(m)\| \leq C\|m\|$  for some universal constant  $C$  and all  $m \in \mathcal{M}_N$ , where  $\|\cdot\|$  denotes any norm on  $\mathcal{M}_N$  or  $\mathbb{R}^N$  respectively.

Function  $\tilde{s} = (\tilde{s}_1, \dots, \tilde{s}_N)$  fulfills assumptions of Itô theorem, hence for every  $T > 0$  we can write the Itô formula

$$\begin{aligned} \tilde{s}_i(A_T) = & \int_0^T \sum_{k,l} \frac{\partial \tilde{s}_i(m)}{\partial m_{kl}} \Big|_{m=A_t} dM_{kl} + \\ & \sum_{k,l} \frac{1}{4N} \left( \frac{\partial^2 \tilde{s}_i(m)}{(\partial \Re m_{kl})^2} \Big|_{m=A_t} + \frac{\partial^2 \tilde{s}_i(m)}{(\partial \Im m_{kl})^2} \Big|_{m=A_t} \right) dt. \end{aligned}$$

For every  $\omega \in \Omega$  such that  $A_t(\omega) \in K_\epsilon$  for all  $0 \leq t \leq T$  the left-hand side of this equation is equal to  $\lambda_i(T)$  and the right-hand side can be

computed from (8):

$$\lambda_i(T) = \int_0^T \Re(dB_{ii}) + \frac{dt}{2\lambda_i} \left( 1 - \frac{1}{2N} + \sum_{j \neq i} \frac{\lambda_i^2 + \lambda_j^2}{N(\lambda_i^2 - \lambda_j^2)} \right).$$

Since almost every  $\omega$  has the property that for some  $\epsilon > 0$  we have  $A_t \in K_\epsilon$  for all  $0 \leq t \leq T$  hence the above equation holds without any restrictions for  $\omega$ . Equivalently,

$$d\lambda_i(t) = \Re(dB_{ii}) + \frac{dt}{2\lambda_i} \left( 1 - \frac{1}{2N} + \sum_{j \neq i} \frac{\lambda_i^2 + \lambda_j^2}{N(\lambda_i^2 - \lambda_j^2)} \right).$$

## 5.2. Determinant of a standard Gaussian random matrix.

**Proposition 9.** *Let  $(G^{(N)})$  be a sequence of independent standard Gaussian random matrices and let  $c$  be a circular element. Then*

$$\lim_{N \rightarrow \infty} \operatorname{tr} \ln |G^{(N)}| = \ln \Delta(c) = -\frac{1}{2}$$

*holds almost surely.*

*Furthermore for any  $s \in \mathbb{R}$  we have*

$$\mathbb{E} \min \left( s, \inf_N \operatorname{tr} \ln |G^{(N)}| \right) > -\infty.$$

*Proof.* The square of a circular element is a free Poisson element with parameter 1. The probability density of this element can be explicitly calculated [VDN] and the integral  $\ln \Delta(c) = \int_0^\infty \ln r \, d\mu_{\sqrt{cc^*}}$  can be computed directly.

Let us fix  $N \in \mathbb{N}$ . Let  $v_1, \dots, v_N$  be random vectors in  $\mathbb{C}^N$  which are defined to be columns of the matrix  $G^{(N)}$ . We define

$$V_i = \sqrt{\det \begin{bmatrix} \langle v_1, v_1 \rangle & \cdots & \langle v_1, v_i \rangle \\ \vdots & & \vdots \\ \langle v_i, v_1 \rangle & \cdots & \langle v_i, v_i \rangle \end{bmatrix}},$$

where  $\langle \cdot, \cdot \rangle$  is the standard hermitian form on  $\mathbb{C}^N$ . The above matrix  $[\langle v_k, v_l \rangle]_{1 \leq k, l \leq i}$  is the complex analogue of the Gram matrix; therefore—informally speaking—we can regard  $V_i$  to be the “complex volume” of the “complex parallelepiped” defined by vectors  $v_1, \dots, v_i$ .

Of course  $V_{i+1}$  is equal to the product of  $V_i$  and  $l_{i+1}$ , where  $l_{i+1}$  is the length of the projection of the vector  $v_{i+1}$  onto the orthogonal complement of the vectors  $v_1, \dots, v_i$ . Since

$$\begin{bmatrix} \langle v_1, v_1 \rangle & \cdots & \langle v_1, v_N \rangle \\ \vdots & & \vdots \\ \langle v_N, v_1 \rangle & \cdots & \langle v_N, v_N \rangle \end{bmatrix} = (G^{(N)})^* G^{(N)}$$

it follows that

$$|\det G^{(N)}| = V_N = l_1 l_2 \cdots l_N.$$

It is easy to see that the distribution of  $l_i$  coincides with the distribution of the length of a random Gaussian vector with an appropriate covariance in the complex  $(N - i + 1)$ -dimensional space and therefore

$$\mathbb{E} l_i^{-h} = \frac{\int_0^\infty r^{-h} r^{2(N-i+1)-1} e^{-Nr^2} dr}{\int_0^\infty r^{2(N-i+1)-1} e^{-Nr^2} dr} = \frac{N^{\frac{h}{2}} \Gamma(N - i + 1 - \frac{h}{2})}{\Gamma(N - i + 1)}$$

and hence Markov inequality gives us

$$P(l_N^{-1} > e^{N\epsilon}) < e^{-N\epsilon} \sqrt{\pi N}$$

$$P[(l_1 \cdots l_{N-1})^{-2} > e^{(1+2\epsilon)N}] < e^{(-1-2\epsilon)N} N^{N-1} \frac{1}{\Gamma(N)} < e^{-2\epsilon N}.$$

Above we have used that random variables  $l_i$  are independent and simple inequality  $\Gamma(N) > (\frac{N-1}{e})^{N-1}$  for  $N \in \mathbb{N}$ .

Since

$$\begin{aligned} P\left(\frac{\log l_1 + \cdots + \log l_N}{N} < -\frac{1}{2} - 2\epsilon\right) &\leq \\ P\left(\frac{\log l_1 + \cdots + \log l_{N-1}}{N} < -\frac{1}{2} - \epsilon\right) &+ P\left(\frac{\log l_N}{N} < -\epsilon\right) \end{aligned}$$

Borel–Cantelli lemma implies

$$\liminf_{N \rightarrow \infty} \operatorname{tr} \ln |G^{(N)}| \geq \ln \Delta(c)$$

almost surely. This together with Lemma 2 gives us the first part of the proposition.

It is possible to find a constant  $C$  such that for every  $\epsilon > \frac{1}{4}$  we have

$$P\left(\inf_N \operatorname{tr} \ln |G^{(N)}| < -\frac{1}{2} - 2\epsilon\right) \leq \sum_N P\left(\operatorname{tr} \ln |G^{(N)}| < -\frac{1}{2} - 2\epsilon\right) \leq C e^{-\epsilon}.$$

If  $\nu$  is the distribution of the random variable  $\inf_N \operatorname{tr} \ln |G^{(N)}|$  then integration by parts gives

$$\int_{-\infty}^{-1} (t+1) d\nu(t) = - \int_{-\infty}^{-1} \nu(-\infty, t) dt > -\infty$$

and the second part follows.  $\square$

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